Estimation of Hawkes processes from binned observations using Whittle likelihood

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Motivation

Study the dynamics of contagious diseases and their transmission with respect to risk factors.

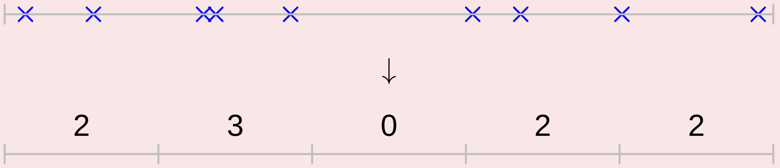
Attributable fraction for contagious diseases

- Autoregressive models may be difficult to interpret in an epidemiological context.
- Potentially rarely occurring events.

$\rightarrow$ Hawkes process (Meyer, Elias, and Höhle, 2012).

Motivation

Problem: aggregate datasets



Other approaches

- (Kirchner, 2016) Convergence of a well-defined INAR(∞) process to a Hawkes process when the binsize converges to 0.
- (Celeux, Chauveau, and Diebolt, 1995) Convergence of the Stochastic EM algorithm?

Our approach inspired from (Adamopoulos, 1976; Roueff and Sachs, 2019): Whittle likelihood for Hawkes bin-count sequences.

Outline

Outline

Point process

Definition: Point process X on $\mathbb{R}$

Random point pattern on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$:

$$\begin{aligned} X : (\Omega, \mathcal{F}, \mathbb{P}) &\to (\mathfrak{X}, \mathcal{X}) \\ \omega &\mapsto \{X_i(\omega)\}_{i \in \mathbb{Z}} \end{aligned}$$

where $\mathfrak{X}$ is the set of locally finite subset of $\mathbb{R}$.

Point process

Definition: Point process N on $\mathbb{R}$

Measurable map N:

$$\begin{aligned} N : (\Omega, \mathcal{F}, \mathbb{P}) &\to (\mathfrak{N}, \mathcal{N}) \\ \omega &\mapsto N(\omega, \cdot) \end{aligned}$$

where $\mathfrak{N}$ is the set of locally finite counting measures on $\mathbb{R}$.

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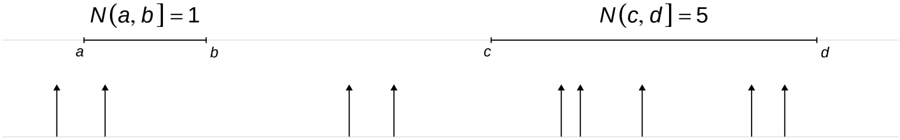


Hawkes process

Conditional intensity λ^* of point process X

$\lambda^*(t)dt$ is the conditional probability that there will be a point of X between t and $t + dt$, given the realisations of X before t:

$$\lambda^*(t)dt = \mathbb{P}(N(dt) > 0 \mid \{t_j\}, t_j < t)$$

Hawkes process on the real half-line (Hawkes, 1971)

Self-exciting point process defined by its conditional intensity function:

$$\lambda^*(t) = \eta(t) + \sum_{t_j < t} h(t - t_j)$$

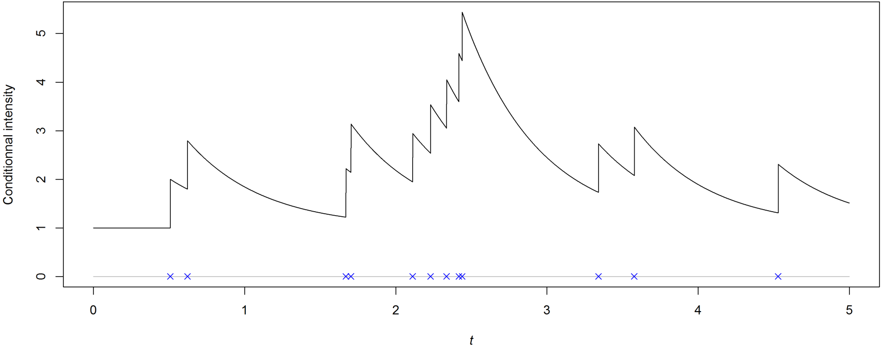
where η, h are integrable nonnegative functions and $(t_j)_{j \in \mathbb{N}}$ are realisations of the point process.

The occurrence of any event increases temporarily the probability of further events occurring.

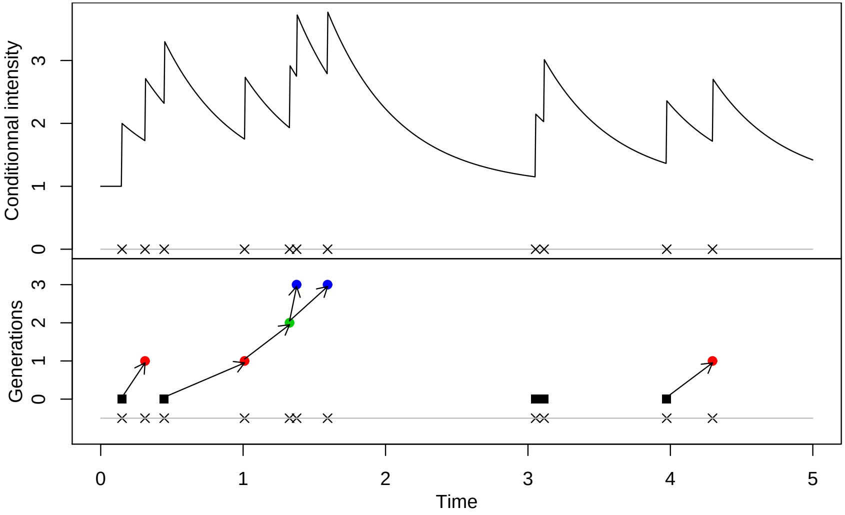
Hawkes process on the real half-line

With exponentially decaying intensity:

$$\lambda^*(t) = \eta + \sum_{t_j < t} \alpha e^{-\beta(t - t_j)}$$



Hawkes process as a branching process



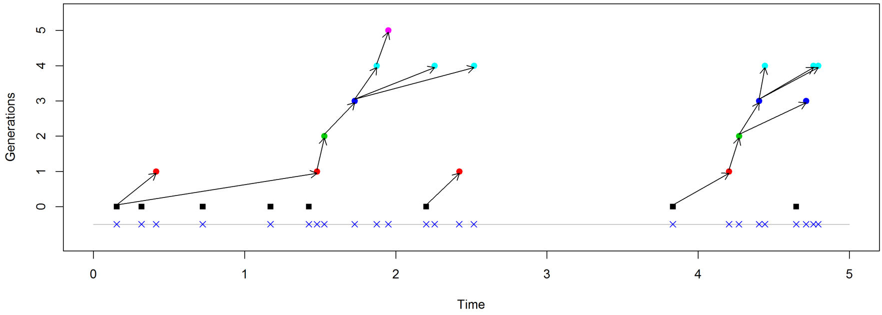
Epidemiological interpretation

Basic reproduction number

Mean number of infections caused by an individual

$$\mu = \int_0^\infty h(t)dt$$

$$= \alpha/\beta \qquad \text{for exponentially decaying intensity}$$



Outline

Spectral representation

Bartlett spectrum (Daley and Vere-Jones, 2008, Proposition 8.2.I)

For a second-order stationary point process N on $\mathbb{R}$, then

$$\mathrm{Cov}\left(N(\varphi), N(\psi)\right) = \int_{\mathbb{R}} \widetilde{\varphi}(\omega)\widetilde{\psi^*}(\omega)\Gamma(d\omega)$$

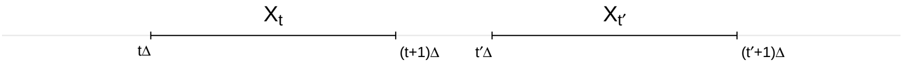
where φ and ψ are functions of rapid decay, $\psi^*(u) = \psi(-u)$, and $\widetilde{\cdot}$ denotes the Fourier transform: $\widetilde{\varphi}(\omega) = \int_{\mathbb{R}} e^{i\omega u}\varphi(u)du$.
The unique measure $\Gamma(\cdot)$ is called the *Bartlett spectrum* of N.

For the stationary Hawkes process N, the Bartlett spectrum admits a density given by (Daley and Vere-Jones, 2008, Example 8.2(e))

$$\gamma(\omega) = \frac{m}{2\pi}|1 - \widetilde{h}(\omega)|^{-2}$$

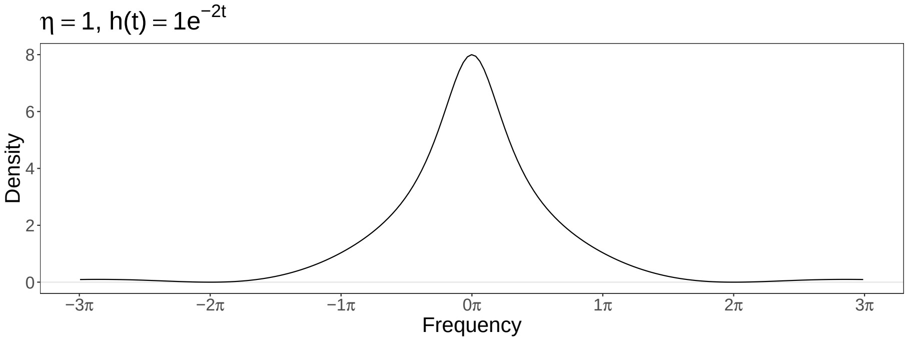
with $m = \mathbb{E}\left[N(0,1]\right] = \eta(1-\mu)^{-1}$.

Spectral representation of the bin-count process

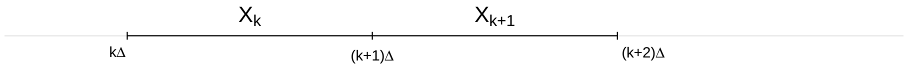


For the Hawkes bin-count process $\{X_t\}_{t\in\mathbb{R}} = \{N(t\Delta,(t+1)\Delta)\}_{t\in\mathbb{R}}$, the spectral density is given by

$$f_{X_t}(\omega) = m\,\Delta\,\mathrm{sinc}^2\left(\frac{\omega}{2}\right)\left|1-\widetilde{h}\left(\frac{\omega}{\Delta}\right)\right|^{-2}$$

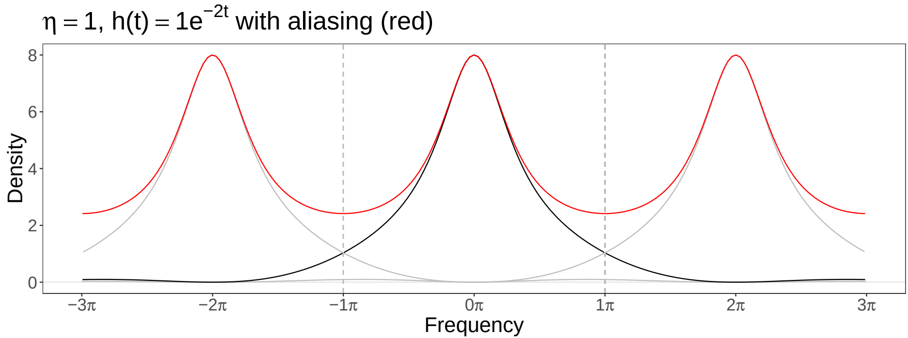


Spectral representation of the bin-count sequence



For the bin-count sequence $\{X_k\}_{k\in\mathbb{Z}} = \{N(k\Delta, (k+1)\Delta)\}_{k\in\mathbb{Z}}$, the spectral density is given by

$$f_{X_k}(\omega) = \sum_{k\in\mathbb{Z}} f_{X_t}(\omega + 2k\pi)$$



The Whittle likelihood (Whittle, 1952)

Consider a bin-count sequence (X_k) with spectral density f_θ. Define

$$\widehat{\theta}_n = \underset{\theta\in\Theta}{\arg\min}\ \mathcal{L}_n(\theta)$$

where $\mathcal{L}_n(\theta)$ is the log-spectral likelihood of the process

$$\mathcal{L}_n(\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(\log f_\theta(\omega) + \frac{I_n(\omega)}{f_\theta(\omega)}\right)\mathrm{d}\omega,$$

$I_n(\omega)$ is the periodogram of (X_k).

Asymptotic properties for $\widehat{\theta}_n$

- For Gaussian* processes (Whittle, 1953);
- For linear processes (Hosoya, 1974; Dzhaparidze, 1974);
- For strongly mixing processes (Dzhaparidze, 1986).

Outline

Weak dependence for time series and random fields

- Rosenblatt's strong-mixing coefficient (1956), to measure the dependence between σ-algebras:

$$\alpha(\mathcal{A},\mathcal{B}) := \sup\Big\{ |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A\in\mathcal{A}, B\in\mathcal{B}\Big\}.$$

- Strong mixing coefficient for a time series $(X_k)_{k\in\mathbb{Z}}$:

$$\alpha_X(r) := \sup_{k\in\mathbb{Z}} \alpha\big(\mathcal{F}_{-\infty}^{k}, \mathcal{F}_{k+r}^{\infty}\big), \qquad \text{where } \mathcal{F}_a^b = \sigma\big(X_k, a\le k\le b\big).$$

- Provides strong moment inequalities and coupling methods (Doukhan, 1994; Rio, 2017), *provided the coefficient decreases fast enough*.
- Other existing mixing coefficients, notably the absolute regularity mixing coefficients.
  - Easily computed for (functions of) Markov processes.

See (Bradley, 2005) for a short review of mixing conditions for time series and random fields.

Mixing properties for point processes

Define, for a Borel set $A \in \mathbb{R}$, the cylindrical σ-algebra generated by a point process N on A:

$$\mathcal{E}(A) \coloneqq \sigma\big(\{N \in \mathfrak{N} : N(B) = m\}, B \in \mathcal{B}(A), m \in \mathbb{N}\big).$$

Strong mixing coefficient for a point process N (Westcott, 1972)

Dependence between past and future events:

$$\alpha_N(r) \coloneqq \sup_{t \in \mathbb{R}} \alpha\big(\mathcal{E}_{-\infty}^{t}, \mathcal{E}_{t+r}^{\infty}\big), \qquad \text{where } \mathcal{E}_a^b = \mathcal{E}\big((a, b]\big).$$

- Poisson cluster processes are mixing (Westcott, 1971).
- Recent applications of mixing coefficients for point processes (Heinrich and Pawlas, 2013; Poinas, Delyon, and Lavancier, 2019).

Strong mixing properties for Hawkes processes

Theorem

Let N be a stationary Hawkes process on $\mathbb{R}$ with reproduction function $h = \mu h^*$, $\mu = \int_{\mathbb{R}} h$. Suppose that there exists a $\delta > 0$ such that the distribution kernel h^* has a finite moment of order $1 + \delta$:

$$\nu_{1+\delta} := \int_{\mathbb{R}} t^{1+\delta} h^*(t) \mathrm{d}t < \infty.$$

Then N is strongly mixing and

$$\alpha_N(r) = \mathcal{O}\left(r^{-\delta}\right).$$

Ideas of the proof

We need to bound

$$\alpha_N(r) \coloneqq \sup_{t\in\mathbb{R}} \alpha\big(\mathcal{E}_{-\infty}^{t}, \mathcal{E}_{t+r}^{\infty}\big) = \sup_{t\in\mathbb{R}} \sup_{\substack{\mathcal{A}\in\mathcal{E}_{-\infty}^{t}\\ \mathcal{B}\in\mathcal{E}_{t+r}^{\infty}}} \left|\mathrm{Cov}\big(\mathbb{1}_{\mathcal{A}}(N), \mathbb{1}_{\mathcal{B}}(N)\big)\right|,$$

where $\mathbb{1}_{\mathcal{A}}(N)$ is the indicator function of the cylinder set $\mathcal{A}$, *i.e.* for an elementary cylinder set $\mathcal{A}_{B,m} = \{N \in \mathfrak{N} : N(B) = m\}$,

$$\mathbb{1}_{\mathcal{A}_{B,m}}(N) = \begin{cases} 1 & \text{if } N(B) = m, \\ 0 & \text{otherwise.} \end{cases}$$

Ideas of the proof (cont'd)

$$\alpha_N(r) = \sup_{t\in\mathbb{R}} \sup_{\substack{\mathcal{A}\in\mathcal{E}_{-\infty}^{t} \\ \mathcal{B}\in\mathcal{E}_{t+r}^{\infty}}} \left|\mathrm{Cov}\left(\mathbb{1}_{\mathcal{A}}(N), \mathbb{1}_{\mathcal{B}}(N)\right)\right| \quad (1)$$

1. Control (1) by the covariance of counts.
    - Hawkes processes are infinitely divisible (Evans, 1990, Theorem 1.1) ;
    - They are positively associated (Gao and Zhu, 2018, Section 2.1, key property (e)) ;
    - Use of Theorem 2.5 from (Poinas, Delyon, and Lavancier, 2019).
2. Rescale to a single branching process by conditioning on the cluster centre process.
3. Control the covariance of counts of a single branching process.
    - Almost sure extinction of the subcritical Galton-Watson tree;
    - Finite moments for the reproduction kernel.
4. Integrate back with respect to the cluster centre process.

Asymptotic properties of the Whittle estimator

Direct application of (Dzhaparidze, 1986, Theorem II.7.1).

Consistency

Let $(X_k)_{k\in\mathbb{Z}} = \big(N(k,k+1]\big)_{k\in\mathbb{Z}}$ be the bin-count sequences of a stationary Hawkes process, with spectral density function f_θ. Assume the following regularity conditions on f_θ:

(A1) The true value θ_0 belongs to a compact set $\Theta \subset \mathbb{R}^p$.
(A2) For all $\theta_1 \neq \theta_2$ in Θ, then $f_{\theta_1} \neq f_{\theta_2}$ for almost all ω.
(A3) The function f_θ^{-1} is differentiable with respect to θ and its derivatives $(\partial/\partial\theta_k) f_\theta^{-1}$ are continuous in $\theta \in \Theta$ and $-\pi \leq \omega \leq \pi$.

Further assume that there exists a $\delta > 0$ such that the reproduction kernel h^* has a finite moment of order $2+\delta$. Then the estimator $\widehat{\theta}_n$ is consistent, *i.e.* $\widehat{\theta}_n \to \theta_0$ in probability.

Asymptotic properties (cont'd)

Direct application of (Dzhaparidze, 1986, Theorem II.7.2).

Asymptotic normality

Let $(X_k)_{k\in\mathbb{Z}} = \big(N(k, k+1]\big)_{k\in\mathbb{Z}}$ be the bin-count sequences of a stationary Hawkes process, with spectral density function f_θ. Assume conditions (A1), (A2), (A3) and:

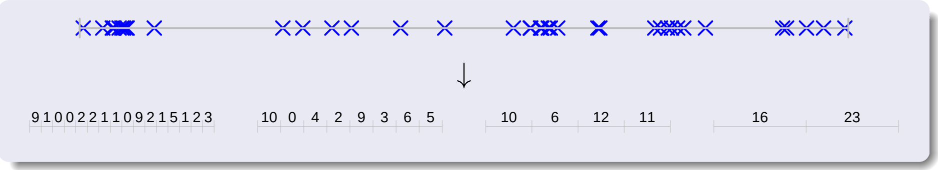
(A4) The function f_θ is twice differentiable with respect to θ and its second derivatives $(\partial^2/\partial\theta_k\partial\theta_l)f_\theta$ are continuous in $\theta \in \Theta$ and $-\pi \leq \omega \leq \pi$.

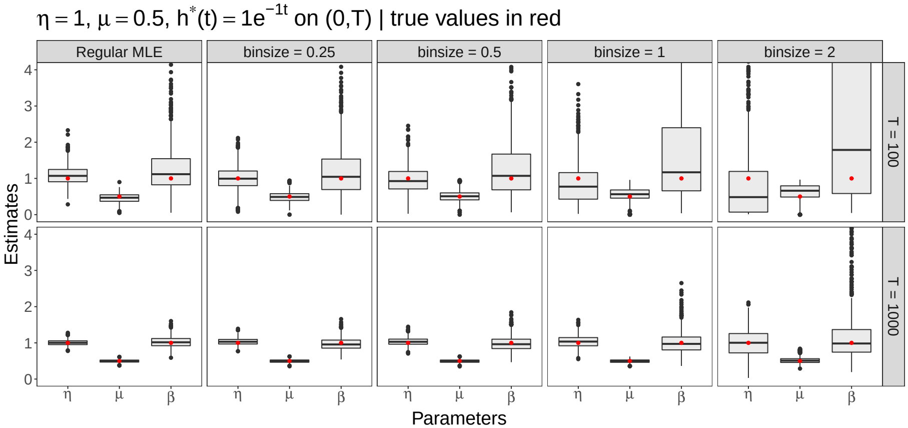
Then the estimator $\widehat{\theta}_n$ is asymptotically normal and

$$n^{1/2}(\widehat{\theta}_n - \theta_0) \underset{n\to\infty}{\sim} \mathcal{N}\left(0, \Gamma_{\theta_0}^{-1} + \Gamma_{\theta_0}^{-1} C_{4,\theta_0} \Gamma_{\theta_0}^{-1}\right).$$

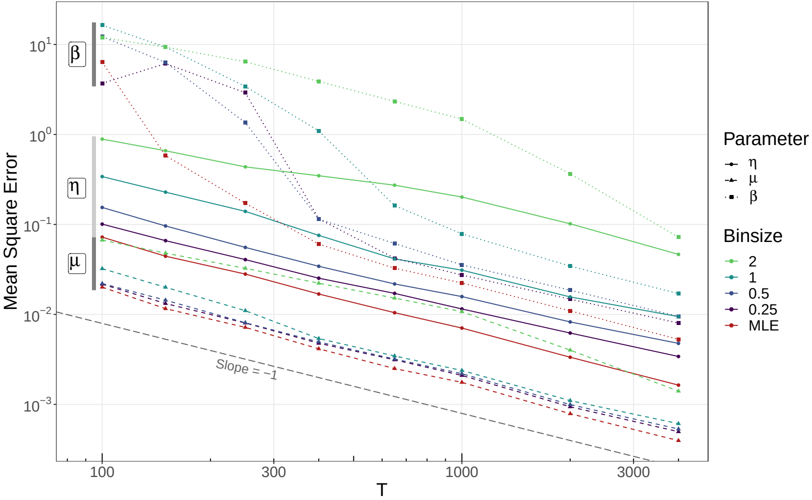
Outline

Simulation for the Whittle estimator

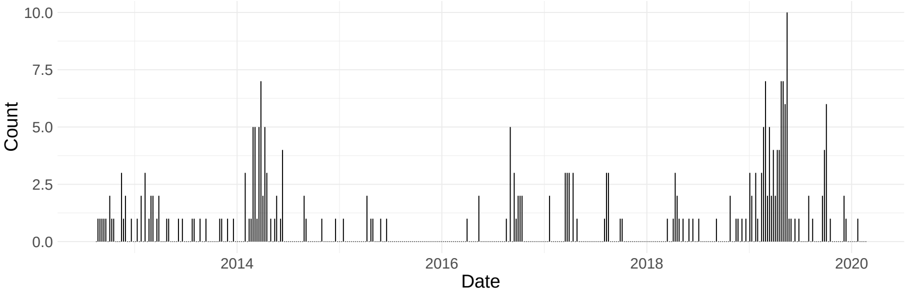




Simulation for the Whittle estimator



Case-study: transmission of Measles in Tokyo[1]



Gaussian reproduction kernel: $h^*(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-\nu)^2}{2\sigma^2}\right)$

- $\widehat{\nu} = 9.8 \text{ days}, \widehat{\sigma} = 5.9 \text{ days}$

Epidemiology (Centers for Disease Control and Prevention, 2015)

Incubation period: 10-12 days after exposure.

Transmission period: 4 days before to 4 days after rash onset.

[1] https://www.niid.go.jp/niid/en/survaillance-data-table-english.html

Outline

Conclusion

- Good asymptotic properties, similar to maximum likelihood estimation;
- Easy to implement and flexible, only need to input $\widetilde{h}$;
- Computationally efficient : $\mathcal{O}(n \log n)$ instead of $\mathcal{O}(n^2)$ for maximum likelihood.

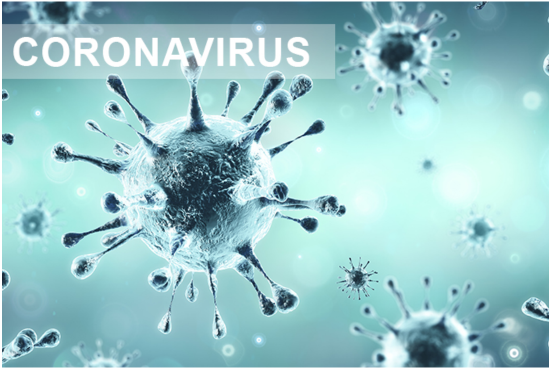
Simulation and estimation methods implemented in R package *hawkesbow*.[2]

Extensions

- **Non causal Hawkes processes**: strong mixing properties are still satisfied.
- **Multivariate Hawkes processes** (work in progress with Ousmane Boly and Thi Hien Nguyen):
  - Multivariate Bartlett spectrum (Daley and Vere-Jones, 2003, Example 8.3(c));
  - Multitype Galton-Watson trees.

[2] https://github.com/fcheysson/hawkesbow

Perspectives



- Non-stationary Hawkes processes: allow all parameters to vary with time and be dependent on explanatory variables;
- Transient explosivity: allow μ to temporarily be higher than 1.

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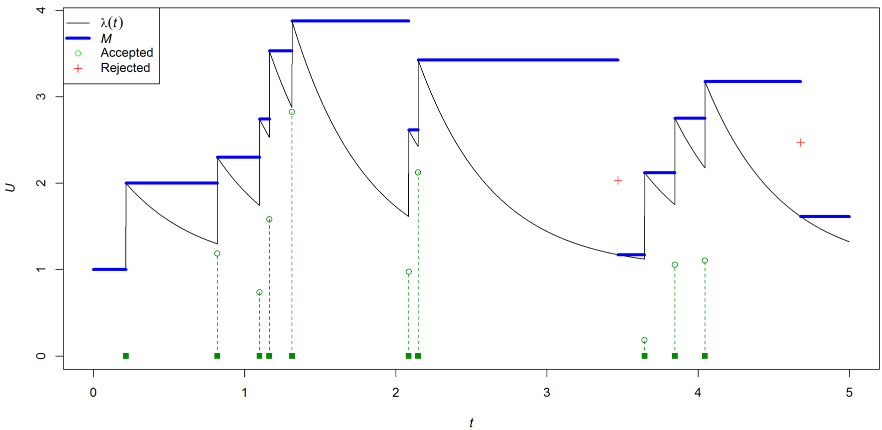
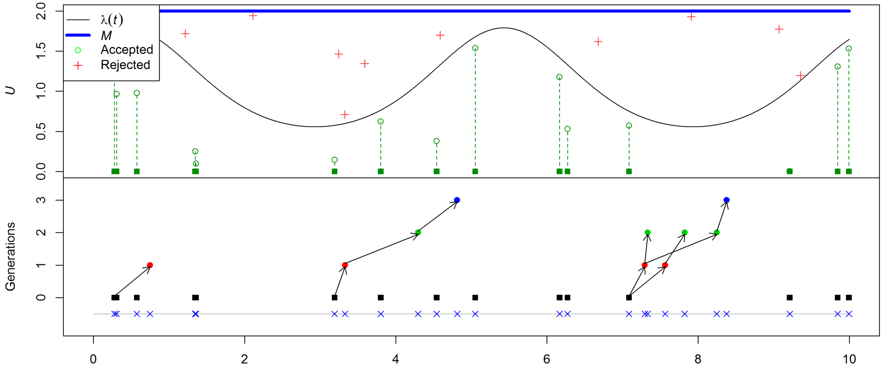
Simulate Hawkes in *R* (Ogata, 1981)

```
sim <- hawkes(T=10, fun=1, repr=1, family=''exp'', rate=2)
plot(sim, intensity = TRUE)
```

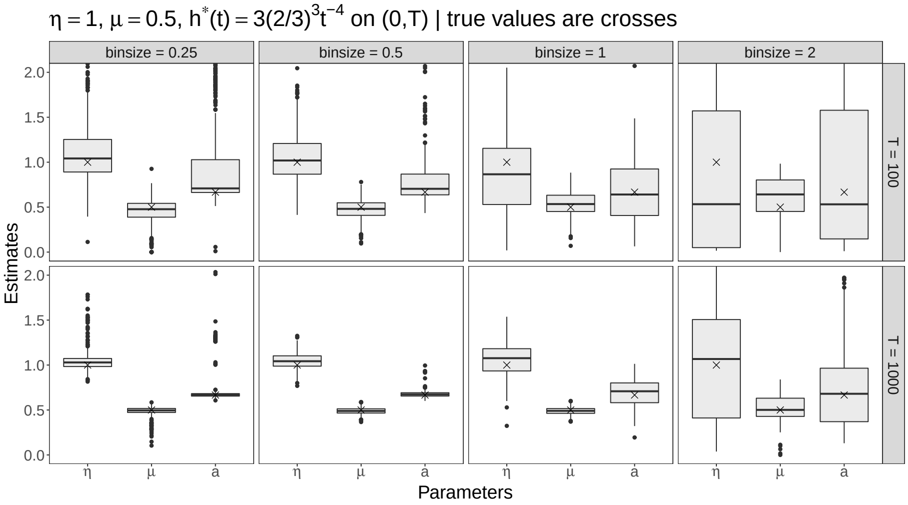


Simulate Hawkes with inhomogeneous background intensity in *R* (Møller and Rasmussen, 2005; Dassios and Zhao, 2013)

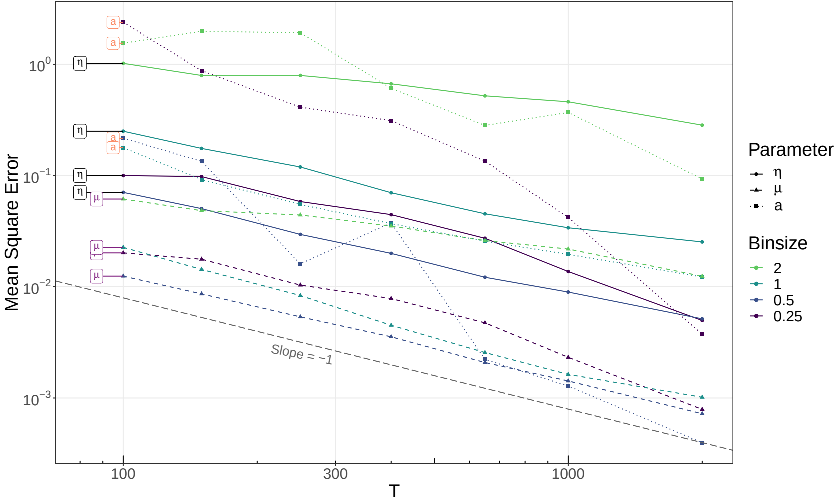
```
int <- function(t) exp(.5*cos(2*pi*t/5)+.3*sin(2*pi*t/5))
sim <- hawkes(T=10, fun=int, M=2, repr=1, family=''exp'', rate=
plot(sim$immigrants)
plot(sim)
```



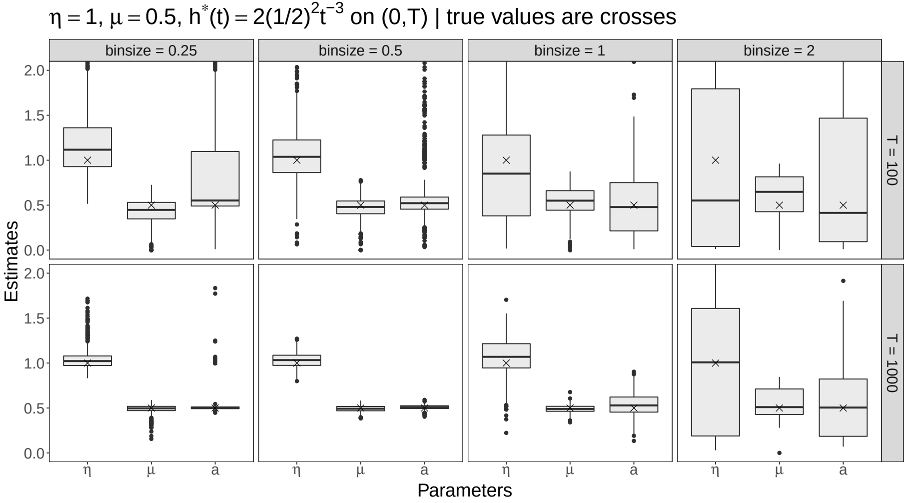
Pareto kernel, $\gamma = 3$



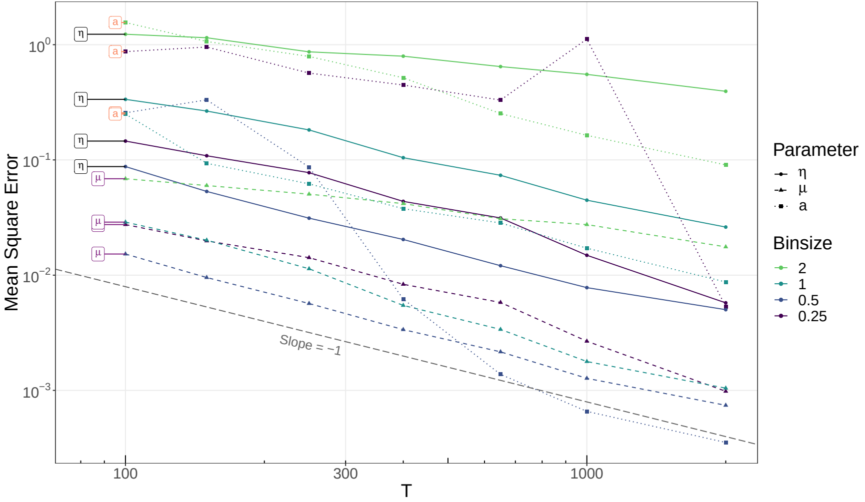
Pareto kernel, $\gamma = 3$



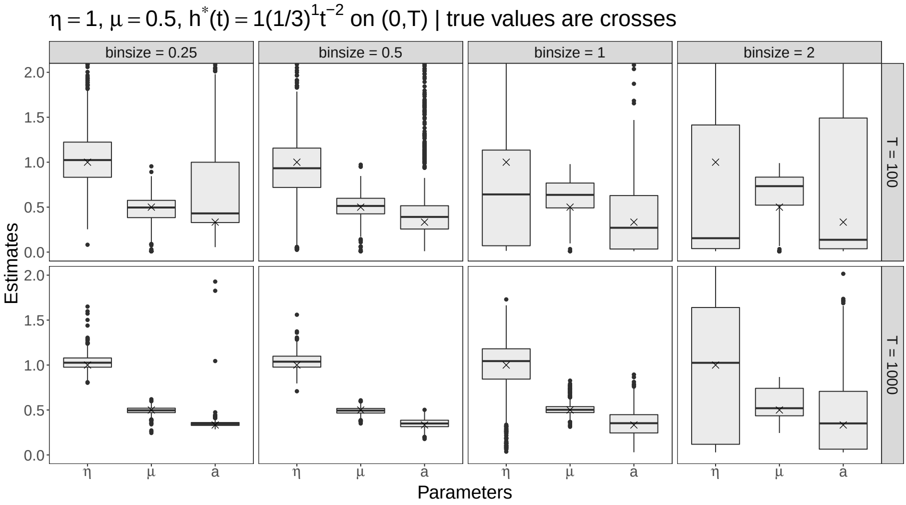
Pareto kernel, $\gamma = 2$



Pareto kernel, $\gamma = 2$



Pareto kernel, $\gamma = 1$



Pareto kernel, $\gamma = 1$